Journal of Statistical Physics, Vol. 94, Nos. 5/6, 1999

Boundary Conditions for Scalar Conservation Laws from a Kinetic Point of View

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Received May 15, 1998

Boundary conditions for multidimensional scalar conservation laws are obtained in the context of hydrodynamic limits from a kinetic point of view. The initial boundary value kinetic problem is well posed since inward and outward characteristics of the domain can be distinguished. The convergence of the first momentum of the distribution function to an entropy solution of the conservation law is established. Boundary conditions are obtained. The equivalence with the Bardos, Leroux, and Nedelec conditions is studied.

KEY WORDS: Hydrodynamic limits; multidimensional scalar conservation laws; kinetic approach; Cauchy problem and boundary conditions; BV estimates.

1. INTRODUCTION

Defining boundary conditions for an initial boundary value problem of general scalar conservation laws is not straightforward, since the ingoing flux depends on the solution to the problem. In ref. 1, Bardos, Leroux, and Nedelec studied the vanishing viscosity limit for the solution to the initial boundary-value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(u_\varepsilon(t,x)) + div_x A(u_\varepsilon(t,x)) = \varepsilon\, \Delta u_\varepsilon, & t>0, \quad x \in \Omega \\ u_\varepsilon(0,x) = u_0(x), & x \in \Omega \\ u_\varepsilon(t,x) = w(t,x), & t>0, \quad x \in \partial\Omega \end{cases} \tag{1}$$

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0022-4715/99/0300-0779$16.00/0

Here Ω is a bounded subset of $\mathbb{R}^N$, $A=(A_i)_{1\leqslant i\leqslant N}$ is a $\mathscr{C}^2$ function, and w is a given data. By a proper choice of test functions introduced by Kruskov,[9] they proved that u_ε converges towards the unique solution u to

$$-\int_\Omega |u_0(x)-\ell|\,\psi(x)\,dx$$

$$-\int_{\mathbb{R}^+}\int_\Omega [\,|u-\ell|\,\partial_t\psi+\operatorname{sgn}(u-\ell)(A(u)-A(\ell))\cdot\nabla_x\psi\,]\,dt\,dx$$

$$+\int_{\mathbb{R}^+}\int_{\partial\Omega}\operatorname{sgn}(w-\ell)(A(u)-A(\ell))\cdot n\psi\,dt\,dx\leqslant 0 \tag{2}$$

for any C^1 test function ψ with compact support in $[0,T]\times\bar{\Omega}$. Here n is the outward normal to $\partial\Omega$. Hence n satisfies an entropy inequality in Ω,

$$\frac{\partial}{\partial t}|u-\ell|+div_x(\operatorname{sgn}(u-\ell)(A(u)-A(\ell)))\leqslant 0,\qquad \ell\in\mathbb{R}$$

together with an entropy inequality in $\partial\Omega$,

$$(\operatorname{sgn}(u-\ell)-\operatorname{sgn}(w-\ell))(A(u)-A(\ell))\cdot n\geqslant 0,\qquad \ell\in\mathbb{R}. \tag{3}$$

We shall refer to this last inequality as the (BLN) boundary condition.

Another approach for defining boundary conditions to scalar conservation laws as well as hyperbolic systems in one dimensional space variable, consists in using solutions to the Riemann problem.[4] Moreover, multidimensional initial boundary value problems with strong linearities are investigated in ref. 10. A number of studies has been made concerning the following kinetic model for scalar conservation laws in $\mathbb{R}^N$,

$$\frac{\partial}{\partial t}f_\varepsilon(t,x,v)+\sum_{i=1}^N a_i(v)\frac{\partial}{\partial x_i}f_\varepsilon(t,x,v)=\frac{1}{\varepsilon}[\chi_{u_{f_\varepsilon}(t,x)}(v)-f_\varepsilon(t,x,v)] \tag{4}$$

where $a_i=A_i'$, $1\leqslant i\leqslant N$ (see refs. 2, 6, 8, 11–13,...). Denote by

$$a(v)=(a_1(v),a_*(v)),\qquad a_*(v)=(a_2(v),\ldots,a_N(v))$$

and

$$u_\varepsilon(t,x)=\int_{\mathbb{R}} f_\varepsilon(t,x,v)\,dv. \tag{5}$$

The unknowns $u_\varepsilon, f_\varepsilon$ are defined for $t \geqslant 0$, $x = (x_1, y) \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$, $v \in \mathbb{R}$, and χ is the signature function

$$\chi_{u_\varepsilon}(v) = \begin{cases} \operatorname{sgn} u_0 & \text{if} \quad (u_\varepsilon - v)\, v \geqslant 0 \\ 0 & \text{else} \end{cases} \tag{6}$$

Notice that

$$u_\varepsilon(t, x) = \int_{-\infty}^{+\infty} \chi_{u_\varepsilon(t,x)}(v)\, dv \tag{7}$$

It is proved in ref. 13 that the first momentum u_ε converges towards the unique entropy solution to

$$\frac{\partial}{\partial t} u(t, x) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(u(t, x)) = 0, \qquad u(0, x) = u_0(x). \tag{8}$$

Here, we investigate the hydrodynamic limit of the solution to the initial boundary value problem on $\Omega := \mathbb{R}_+ \times \mathbb{R}^{N-1}$ with a kinetic approach. We study the solutions to

$$\begin{gathered} \frac{\partial}{\partial t} f_\varepsilon(t, x, v) + \sum_{i=1}^{N} a_i(v) \frac{\partial}{\partial x_i} f_\varepsilon(t, x, v) = \frac{1}{\varepsilon} [\chi_{u_{f_\varepsilon}(t,x)}(v) - f_\varepsilon(t, x, v)] \\ t \in (0, T), \qquad x \in \Omega, \quad v \in \mathbb{R} \\ f_\varepsilon(0, x, v) = f_0(x, v), \qquad x \in \Omega, \quad v \in \mathbb{R} \\ f_\varepsilon(t, (0, y), v) = \tilde{f}(t, y, v), \qquad t \in (0, T), \quad ((0, y), v) \in \Gamma^- \end{gathered} \tag{9}$$

where

$$\Gamma = \{0\} \times \mathbb{R}^{N-1} \times \mathbb{R}_v = \partial\Omega \times \mathbb{R}_v, \qquad \Gamma^- = \{(x, v) \in \Gamma;\, a(v) \cdot n(x) < 0\}.$$

We prescribe the boundary condition only on the inflow part at $x_1 = 0$ as it is usual for linear transport equations. In a setting of hydrodynamic limits, we obtain a set of admissible boundary states for

$$\lim_{\varepsilon \to 0} \int f_\varepsilon(t, (0, y), v)\, dv.$$

The main result of the paper is the following

Theorem 1. Under some technical assumptions (of Proposition 4), the function $u_\varepsilon(t, x) = \int_{\mathbb{R}} f_\varepsilon(t, x, v)\, dv$ converges (up to a subsequence) in

$L^\infty(0, T; L^1_{\ell oc}(\Omega))$ to a function $u \in L^\infty(0, T; BV(\Omega))$ which is a weak solution of the scalar conservation law

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(u) = 0, \qquad (t, x) \in (0, T) \times \Omega$$

$$u(0, x) = u_0(x), \qquad x \in \Omega$$

in the sense that for all nonnegative function $\psi \in \mathscr{C}^1_0([0, T] \times \Omega)$,

$$-\int_0^T \int_\Omega |u - \ell| \, \partial_t \psi + \operatorname{sgn}(u - \ell)(A(u) - A(\ell)) \cdot \nabla_x \psi \, dx \, dt$$

$$+ \int_0^T \int_{\Gamma^-} a(v) \cdot n \, |\tilde{f} - \chi_\ell| \, \psi \, dv \, dx \, dt \leqslant 0.$$

In particular, u satisfies the following entropy inequality on the boundary $\partial\Omega$,

$$\operatorname{sgn}(u - \ell)(A(u) - A(\ell)) \cdot n - \int_{a(v) \cdot n < 0} a(v) \cdot n \, |\tilde{f} - \chi_\ell| \, dv \geqslant 0, \qquad \ell \in \mathbb{R}.$$

If the boundary data $\tilde{f}$ is at equilibrium, i.e., $\tilde{f}(t, x, v) := \chi_{w(t, x)}(v)$ for some function w, this condition becomes

$$\operatorname{sgn}(u - \ell)(A(u) - A(\ell)) \cdot n$$

$$- \operatorname{sgn}(w - \ell)((A \cdot n)^- (w) - (A \cdot n)^- (\ell) \geqslant 0, \qquad \ell \in \mathbb{R}$$

which is the (BLN) condition for the data w.

For some $\tilde{f}$ not at equilibrium, the boundary condition for u is the (BLN) condition for $\tilde{w}$ defined by

$$\int_{a(v) \cdot n(x) < 0} a(v) \cdot n(x) \, \chi_{\tilde{w}}(v) \, dv = \int_{a(v) \cdot n(x) < 0} a(v) \cdot n(x) \, \tilde{f}(v) \, dv.$$

Finally, u is unique in both cases.

Here

$$(A \cdot n)^- (x) - (A \cdot n)^- (y) := \int_y^x \min(a(v) \cdot n, 0) \, dv, \qquad (x, y) \in \mathbb{R}^2.$$

Let us briefly explain our method. First, we prove the well posedness of (9) and state BV estimates of the solution. Then we study the hydrodynamic limit and prove that it satisfies the scalar conservation law together with an entropy inequality. Finally, in Sections 4 and 5, we study the boundary condition, first in the case of equilibrium, then in a specific case of boundary data out of equilibrium.

2. THE KINETIC PROBLEM

The following study is closely related to the work by Perthane and Tadmor.[13] However, the boundary condition must be studied specifically. We define the sets Q, Q_- and Q_+ respectively by

$$Q = \Omega \times \mathbb{R}_v, \qquad Q_- = \{(t, x, v) \in (0, T) \times Q;\, x_1 - ta_1(v) < 0\}$$

$$Q_+ = \{(t, x, v) \in (0, T) \times Q;\, x_1 - ta_1(v) > 0\}.$$

Let $L^1_{a_1}(\Gamma^-)$ be the set of integrable functions on Γ^- with the weight $a_1(v)$.

Theorem 2. If

$$f_0 \in L^1(Q), \qquad \tilde{f} \in L^\infty(0, T; L^1_{a_1}(\Gamma^-))$$

then the problem

$$\begin{cases} \partial_t f_\varepsilon + a(v) \cdot \nabla_x f_\varepsilon = \dfrac{1}{\varepsilon}\, [\chi_{u_\varepsilon} - f_\varepsilon], \quad t \in (0, T),\ x_1 > 0,\ y \in \mathbb{R}^{N-1},\ v \in \mathbb{R} \\ f_\varepsilon(t, (0, y), v) = \tilde{f}(t, y, v), \quad t \in (0, T), \quad y \in \mathbb{R}^{N-1}, \quad a_1(v) > 0 \\ f_\varepsilon(0, x, v) = f_0(x, v) \end{cases} \tag{10}$$

has a unique solution in $L^\infty(0, T; L^1(Q))$. This solution satisfies

$$\begin{aligned} f_\varepsilon(t, x, v) &= f_0(x - ta(v), v)\, e^{-t/\varepsilon} \\ &\quad + \frac{1}{\varepsilon} \int_0^t \chi_{u_\varepsilon(s,\, x + (s-t)\, a(v))}(v)\, e^{(s-t)/\varepsilon}\, ds, \qquad (t, x, v) \in Q_+ \\ f_\varepsilon(t, x, v) &= e^{-x_1/\varepsilon a_1(v)}\, \tilde{f}\left(t - \frac{x_1}{a_1(v)},\, y - \frac{x_1}{a_1(v)}\, a_*(v),\, v\right) \\ &\quad + \frac{1}{\varepsilon} \int_{t - x_1/a_1(v)}^t \chi_{u_\varepsilon(s,\, x + (s-t)\, a(v))}(v)\, e^{(s-t)/\varepsilon}\, ds, \qquad (t, x, v) \in Q_-. \end{aligned} \tag{11}$$

Moreover, if $f_0 \geqslant 0$ and $\tilde{f} \geqslant 0$, then $f_\varepsilon \geqslant 0$ almost everywhere.

Let f_ε and g_ε be two solutions of (10) corresponding respectively to the data $(f_0, \tilde{f})$ and $(g_0, \tilde{g})$. Then for almost every $t \in (0, T)$,

$$\|f_\varepsilon(t, \cdot, \cdot) - g_\varepsilon(t, \cdot, \cdot)\|_{L^1(Q)} \\ \leqslant (e+1)\left(\|f_0 - g_0\|_{L^1(Q)} + \int_0^t \int_{\Gamma^-} |\tilde{f}(\tau, y, v) - \tilde{g}(\tau, y, v)|\, a_1(v)\, d\tau\, dy\, dv \right). \tag{12}$$

The proof of Theorem 2 is easy. It relies on a Banach fixed point argument in $L^\infty(0, T; L^1(Q))$.

3. A PRIORI ESTIMATES

3.1. L^∞ and L^1 Bounds

Proposition 3. (i) If $f_0 \in L^\infty(Q)$ and $\tilde{f} \in L^\infty((0, T) \times \Gamma^-)$, then $f_\varepsilon \in L^\infty((0, T) \times \Omega \times \mathbb{R}_v)$ and

$$\|f_\varepsilon\|_{L^\infty} \leqslant \max(\|f_0\|_{L^\infty}, \|\tilde{f}\|_\infty) + 1.$$

(ii) If $f_0 \in L^1(Q)$, and $\tilde{f} \in L^1(0, T; L^1_{a_1}(\Gamma^-))$, then $u_\varepsilon \in L^\infty(0, T; L^1(\Omega))$ and

$$\|u_\varepsilon\|_{L^\infty(0, T; L^1(\Omega))} \leqslant 2(\|f_0\|_{L^1(Q)} + \|\tilde{f}\|_{L^1(0, T; L^1_{a_1}(\Gamma^-))}).$$

(iii) If

$$\sup_{(t, x) \in (0, T) \times \Omega} \int_{x_1 - ta_1(v) > 0} |f_0(x - ta(v), v)|\, dv < \infty \tag{13}$$

and

$$\sup_{(t, x) \in (0, T) \times \Omega} \int_{x_1 - ta_1(v) < 0} \left| \tilde{f}\left(t - \frac{x_1}{a_1(v)}, y - \frac{x_1}{a_1(v)} a_*(v), v \right) \right| dv < \infty \tag{14}$$

then (u_ε) is uniformly bounded in $L^\infty((0, T) \times \Omega)$.

Proof of Proposition 3. (i) relies on explicit computations and a Gronwall argument.

(ii) Denote by $U_\varepsilon(t) := \int_\Omega u_\varepsilon(t, x)\, dx$. Then

$$
\begin{aligned}
U_\varepsilon(t) &\leqslant e^{-t/\varepsilon}\, \|f_0\|_{L^1(Q)} + \int_{x_1 - ta_1(v) < 0} e^{-x_1/a_1(v)} \\
&\quad \times \left| \tilde{f}\left(t - \frac{x_1}{a_1(v)}, y - \frac{x_1}{a_1(v)}\, a_*(v), v\right)\right| dv\, dx + \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} U_\varepsilon(s)\, ds \\
&\leqslant e^{-t/\varepsilon}\, \|f_0\|_{L^1(Q)} + \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} \int_{\Gamma^-} |\tilde{f}(s, Y, v)|\, a_1(v)\, dY\, dv\, ds \\
&\quad + \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} U_\varepsilon(s)\, ds
\end{aligned}
$$

by the change of variables $(x_1, y) \to (s := t - x_1/a_1(v), Y := y - (x_1/a_1(v))\, a_*(v))$ in the second integral. (ii) follows by a Gronwall argument.

(iii) Denote by

$$
\begin{aligned}
A(t, x) := &\{(s, v) \in (0, T) \times \mathbb{R};\, x_1 - ta_1(v) > 0,\, 0 < s < t\} \\
&\cup \left\{(s, v) \in (0, T) \times \mathbb{R};\, x_1 - ta_1(v) < 0,\, t - \frac{x_1}{a_1(v)} < s < t\right\}
\end{aligned}
$$

Then

$$
\begin{aligned}
|u_\varepsilon(t, x)| \leqslant &\int_{A(t, x)} |\chi_{u_\varepsilon(s,\, x + (s-t)\, a(v))}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dv \\
&+ \int_{x_1 - ta_1(v) > 0} |f_0(x - ta(v), v)|\, dv \\
&+ \int_{x_1 - ta_1(v) < 0} \left| \tilde{f}\left(t - \frac{x_1}{a_1(v)}, y - \frac{x_1}{a_1(v)}\, a_*(v), v\right)\right| e^{-x_1/\varepsilon a_1(v)}\, dv
\end{aligned}
$$

Denote by $V_\varepsilon(t) := \|u_\varepsilon(t, \cdot)\|_{L^\infty(\Omega)}$. Then

$$
V_\varepsilon(t) \leqslant \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} V_\varepsilon(s)\, ds + c
$$

by Assumptions (14) and (15). And so, (ii) follows by a Gronwall argument.

3.2. Finite Speed of Propagation

Let us assume that f_0 and $\tilde{f}$ are compactly supported in v and that Assumptions (14) and (15) hold. Denote by $u_\infty := \sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^\infty((0, T) \times \Omega)}$.

The function χ_{u_ε} in the right-hand side of (11) is supported in $[-u_\infty, u_\infty]$. It then follows that f_ε given by (11) has its support included for all $t \in (0, T)$ in

$$K := \operatorname{supp}_v(f_0) \cup \operatorname{supp}_v(\tilde{f}) \cup [-u_\infty, u_\infty].$$

Denote by

$$a_\infty := \sup_{1 \leqslant i \leqslant N,\, v \in K} |a_i(v)|.$$

Then

$$a_\infty \geqslant \sup_{1 \leqslant i \leqslant N,\, v \in K'} |a_i(v)| \tag{15}$$

where

$$K' = \{v \in \operatorname{supp}_v(f_\varepsilon(t, x, \cdot), (t, x) \in (0, T) \times \mathbb{R}_+ \times \mathbb{R}^{N-1}\}.$$

3.3. BV Estimates

Proposition 4. Assume that (14) and (15) hold and that f_0 and $\tilde{f}$ are functions compactly supported in v, such that

$$\begin{aligned} &f_0 \in L^1(\mathbb{R}_v; BV(\Omega)) \\ &\tilde{f} \in (L^\infty \cap BV)((0, T) \times \mathbb{R}_+ \times \mathbb{R}^{N-1}; L^1_{a_1}(\{v; a_1(v) > 0\})). \end{aligned} \tag{H1}$$

Assume moreover that

$$\int_{0 < x_1 < h,\, v \in \mathbb{R}} |f_0(x, v)|\, dx\, dv \leqslant ch, \qquad h \in (0, 1) \tag{H2}$$

$$\int_{x_1 > 0,\, x_1 - ha_1(v) > 0} |f_0(x - ha(v), v) - f_0(x, v)|\, dx\, dv \leqslant ch, \qquad h \in (0, 1) \tag{H3}$$

$$\int_{a_1(v) > 0,\, 0 < s < h/a_1(v)} |\tilde{f}(s, y, v)|\, a_1(v)\, ds\, dy\, dv \leqslant ch, \qquad h \in (0, 1) \tag{H4}$$

$$\begin{aligned} &\int_{a_1(v) > 0,\, h/a_1(v) < s < t} \left| \tilde{f}\left(s - \frac{h}{a_1(v)}, y - \frac{h}{a_1(v)} a_*(v), v\right) - \tilde{f}(s, y, v) \right| \\ &\qquad \times a_1(v)\, ds\, dy\, dv \leqslant ch, \qquad h \in (0, 1). \end{aligned} \tag{H5}$$

Then (f_ε) is uniformly bounded in $BV((0, T) \times \Omega; L^1(\mathbb{R}_v))$.

Proof of Proposition 4. First Step. Let us first consider the case of the translation along x_j, $j\neq 1$. For $g_\varepsilon(t, x_1, y, v) := f_\varepsilon(t, x_1, y+he_j, v)$, $j\geqslant 2$ in (12), we obtain

$$\int_Q |f_\varepsilon(t, x_1, y+he_j, v) - f_\varepsilon(t, x_1, y, v)|\, dx\, dv$$

$$\leqslant (e+1)\left(\int_Q |f_0(x_1, y+he_j, v) - f_0(x_1, y, v)|\, dx\, dv\right.$$

$$\left. + \int_0^t \int_{\Gamma^-} |\tilde{f}(\tau, y+he_j, v) - \tilde{f}(\tau, y, v)|\, a_1(v)\, d\tau\, dy\, dv\right)$$

so that

$$\left\| \frac{\partial f_\varepsilon}{\partial x_j}(t, \cdot, \cdot)\right\|_{M(\Omega\times\mathbb{R}_v)} \leqslant (e+1)\left(\left\|\frac{\partial f_0}{\partial x_j}\right\|_{M(\Omega\times\mathbb{R}_v)} + \left\|\frac{\partial \tilde{f}}{\partial x_j}\right\|_{M((0,T)\times\Gamma^-_{a_1})}\right).$$

Here, $M(X)$ denotes the set of bounded measures on X.

Second Step. Let us consider the case of the x_1 translation. For the sake of simplicity, restrict to the case where $N=1$, so that f_ε depends on one space variable $x\in\mathbb{R}_+$. Let $h>0$ be given. By the analytic expression (11) of f_ε,

$$\int |f_\varepsilon(t, x+h, v) - f_\varepsilon(t, x, v)|\, dx\, dv$$

$$\leqslant \int_A |\chi_{u_\varepsilon(s, x+h+(s-t)\,a(v))}(v) - \chi_{u_\varepsilon(s, x+(s-t)\,a(v))}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$+ I_1 + \cdots + I_7$$

where

$$A := \{x>0,\ x-ta(v)>0,\ 0<s<t)\}$$

$$\cup \left\{x+h-ta(v)<0,\ t-\frac{x}{a(v)}<s<t\right\}$$

$$\cup \left\{-h<x-ta(v)<0,\ t-\frac{x}{a(v)}<s<t\right\}$$

$$I_1 := \int_{x+h-ta(v)<0,\ t-(x+h)/a(v)<s<t} |\chi_{u_\varepsilon(s, x+h+(s-t)\,a(v))}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$I_2 := \int_{-h<x-ta(v)<0,\, 0<s<t-s/a(v)} |\chi_{u_\varepsilon(s,\, x+h+(s-t)\,a(v))}(v)| \,\frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$I_3 := \int_{x-ta(v)>0} |f_0(x+h-ta(v),\, v) - f_0(x-ta(v),\, v)|\, e^{-t/\varepsilon}\, dx\, dv$$

$$I_4 := \int_{x+h-ta(v)<0} \left|\tilde{f}\left(t-\frac{x+h}{a(v)},\, v\right) - \tilde{f}\left(t-\frac{x}{a(v)},\, v\right)\right| e^{-(x+h)/\varepsilon a(v)}\, dx\, dv$$

$$I_5 := \int_{x+h-ta(v)<0} \left|\tilde{f}\left(t-\frac{x}{a(v)},\, v\right)\right| (e^{-x/\varepsilon a(v)} - e^{-(x+h)/\varepsilon a(v)})\, dx\, dv$$

$$I_6 := \int_{-h<x-ta(v)<0} |f_0(x+h-ta(v),\, v)|\, dx\, dv$$

$$I_7 := \int_{-h<x-ta(v)<0} \left|\tilde{f}\left(t-\frac{x}{a(v)},\, v\right)\right| e^{-x/\varepsilon a(v)}\, dx\, dv.$$

By the change of variables $x \to X := x + (s-t)\, a(v)$,

$$\begin{aligned}
&\int_A |\chi_{u_\varepsilon(s,\, x+h+(s-t)\,a(v))}(v) - \chi_{u_\varepsilon(s,\, x+(s-t)\,a(v))}(v)| \,\frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv \\
&\qquad \leqslant \int_B |\chi_{u_\varepsilon(s,\, X+h)}(v) - \chi_{u_\varepsilon(s,\, X)}(v)| \,\frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX\, dv
\end{aligned}$$

with

$$\begin{aligned}
B := &\{X>0,\, X-sa(v)>0,\, 0<s<t\} \\
&\cup \{X+h-sa(v)<0,\, X>0,\, 0<s<t\} \\
&\cup \{X+h-sa(v)>0,\, X-sa(v)<0,\, 0<s<t\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_A |\chi_{u_\varepsilon(s,\, x+h+(s-t)\,a(v))}(v) - \chi_{u_\varepsilon(s,\, x+(s-t)\,a(v))}(v)| \,\frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv \\
&\qquad \leqslant \int_{0<s<t,\, X\geqslant 0,\, v\in\mathbb{R}} |\chi_{u_\varepsilon(s,\, X+h)}(v) - \chi_{u_\varepsilon(s,\, X)}(v)| \,\frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX\, dv \\
&\qquad = \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} \int_{X\geqslant 0} |u_\varepsilon(s,\, X+h) - u_\varepsilon(s,\, X)|\, dX\, ds \\
&\qquad \leqslant \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, |f_\varepsilon(s,\, x+h,\, v) - f_\varepsilon(s,\, x,\, v)|\, dx\, dv\, ds.
\end{aligned}$$

Then, by the change of variables $x \to X := x + (s-t)\, a(v)$,

$$I_1 \leqslant \int_{X - sa(v) > 0,\, a(v) > 0,\, 0 < X < h,\, 0 < s < t} |\chi_{u_\varepsilon(s,\, X)}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX\, dv$$

$$\leqslant \int_{0 < X < h,\, 0 < s < t} |u_\varepsilon(s, X)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX \leqslant h\, \|u_\varepsilon\|_{L^\infty((0,\, T) \times \Omega)}.$$

Then, by the change of variables $x \to X := x + h + (s-t)\, a(v)$,

$$I_2 \leqslant \int_{0 < X - sa(v) < h,\, 0 < s < t,\, X < h} |\chi_{u_\varepsilon(s,\, X)}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX\, dv$$

$$\leqslant \int_{0 < X < h,\, 0 < s < t} |u_\varepsilon(s, X)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dX \leqslant h\, \|u_\varepsilon\|_{L^\infty((0,\, T) \times \Omega)}.$$

Then, by the change of variables $x \to X := x - ta(v)$,

$$I_3 \leqslant \int_{X > 0,\, v \in \mathbb{R}} |f_0(X + h, v) - f_0(XC, v)|\, dX\, dv \leqslant ch$$

since f_0 belongs to $L^1\ (\mathbb{R}_v; BV(\Omega))$. Then, by the change of variables $x \to s := t - x/a(v)$,

$$I_4 \leqslant \int_{a(v) > 0,\, h/a(v) < s < t} \left| \tilde{f}\left(s - \frac{h}{a(v)}, v\right) - \tilde{f}(s, v) \right| a(v)\, ds\, dv \leqslant ch$$

by (H5). Then by the same change of variables as for I_4,

$$I_5 = \int_{h - sa(v) < 0} |\tilde{f}(s, v)|\, e^{(s-t)/\varepsilon}(1 - e^{-h/\varepsilon a(v)})\, a(v)\, ds\, dv$$

$$\leqslant h \int_{h - sa(v) < 0,\, 0 < s < t} |\tilde{f}(s, v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dv \leqslant h\, \|\tilde{f}\|_{L^\infty(0,\, T;\, L^1_{a_1}(\Gamma^-))}$$

Then, by the change of variables $x \to X := x + h - ta(v)$,

$$I_6 = \int_{0 < X < h,\, v \in \mathbb{R}} |f_0(X, v)|\, dX\, dv < ch$$

by (H2). Then, by the same change of variables as for I_4,

$$I_7 = \int_{a(v)>0,\, 0<s<h/a(v)} |\tilde{f}(s, v)|\, a(v)\, ds\, dv \leqslant ch$$

by (H4). And so,

$$\int |f_\varepsilon(t, x+h, v) - f_\varepsilon(t, x, v)|\, dx\, dv$$

$$\leqslant ch + \int_0^t \frac{1}{\varepsilon} e^{(s-t)/\varepsilon} \int |f_\varepsilon(s, x+h, v) - f_\varepsilon(s, x, v)|\, dx\, dv\, ds$$

which proves that

$$\int |f_\varepsilon(t, x+h, v) - f_\varepsilon(t, x, v)|\, dx\, dv \leqslant ch, \qquad t \in (0, T)$$

by a Gronwall argument.

Third Step. The proof of the uniform boundedness of (f_ε) in $BV((0, T); L^1(\Omega \times \mathbb{R}_v))$ is analogous to the previous proof. Indeed, for $h > 0$,

$$\int_0^{T-h} \int_{x \geqslant 0,\, v \in \mathbb{R}} |f_\varepsilon(t+h, x, v) - f_\varepsilon(t, x, v)|\, dt\, dx\, dv \leqslant ch$$

by bounding from above $\int_{x \geqslant 0,\, v \in \mathbb{R}} |f_\varepsilon(t+h, x, v) - f_\varepsilon(t, x, v)|\, dx\, dv$ by

$$\int_C |\chi_{u_\varepsilon(s+h,\, x+(s-t)\, a(v))}(v) - \chi_{u_\varepsilon(s,\, x+(s-t)\, a(v))}(v)|\, \frac{1}{\varepsilon} e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$+ J_1 + \cdots + J_7$$

where

$$C := \{x - ta(v) > 0,\, x - (t+h)\, a(v) > 0,\, 0 < s < t\}$$

$$\cup \left\{x - ta(v) < 0,\, t - \frac{x}{a(v)} < s < t\right\}$$

$$\cup \{0 < x < ta(v)ha(v),\, 0 < st\}$$

and

$$J_1 := \int_{x - ta(v) > 0,\ x - (t+h)\, a(v) > 0} |f_0(x - (t+h)\, a(v), v) - f_0(x - ta(v), v)|$$

$$\times e^{-(t+h)/\varepsilon}\, dx\, dv$$

$$J_2 := (1 - e^{-h/\varepsilon})\, e^{-t/\varepsilon} \int_{x - ta(v) > 0,\ x - (t+h)\, a(v) > 0} |f_0(x - ta(v), v)|\, dx\, dv$$

$$J_3 := \int_{x - ta(v) < 0} \left| \tilde{f}\left(t + h - \frac{x}{a(v)}, v\right) - \tilde{f}\left(t - \frac{x}{a(v)}, v\right)\right| e^{-x/\varepsilon a(v)}\, dx\, dv$$

$$J_4 := \int_{0 < x - ta(v) < ha(v)} |f_0(x - ta(v), v)|\, dx\, dv$$

$$J_5 := \int_{0 < x - ta(v) < ha(v)} \left| \tilde{f}\left(t + h - \frac{x}{a(v)}, v\right)\right| e^{-x/\varepsilon a(v)}\, dx\, dv$$

$$J_6 := \int_{x - ta(v) > 0,\ x - (t+h)\, a(v) > 0,\ -h < s < 0} |\chi_{u_\varepsilon(s+h,\, x + (s-t)\, a(v))}(v)|$$

$$\times \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$J_7 := {}_{0 < x - ta(v) < ha(v),\ 0 < s < t - x/a(v)} |\chi_{u_\varepsilon(s,\, x + (s-t)\, a(v))}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv.$$

Then, by the change of variables $x \to X := x + (s-t)\, a(v)$,

$$\int_C |\chi_{u_\varepsilon(s+h,\, x+(s-t)\, a(v))}(v) - \chi_{u_\varepsilon(s,\, x+(s-t)\, a(v))}(v)|\, \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon}\, ds\, dx\, dv$$

$$\leqslant \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} \int_{X \geqslant 0} |u_\varepsilon(s+h, X) - u_\varepsilon(s, X)|\, dX\, ds$$

$$\leqslant \int_0^t \frac{1}{\varepsilon}\, e^{(s-t)/\varepsilon} \int_{X \geqslant 0,\ v \in \mathbb{R}} |f_\varepsilon(s+h, X, v) - f_\varepsilon(s, X, v)|\, dX\, dv\, ds.$$

Moreover,

$$J_1 + J_3 + J_4 + J_5 \leqslant ch$$

by assumption, and

$$J_2 + J_6 + J_7 \leqslant c\,\frac{1}{\varepsilon}\,e^{-t/\varepsilon}h$$

so that

$$\int_0^T (J_2 + J_6 + J_7)(t)\,dt \leqslant ch.$$

4. HYDRODYNAMIC LIMITS

This section is devoted to the first part of the proof of Theorem 1, stated in the introduction. The following proposition and lemma are preliminarily proven.

Proposition 5. Assume that the assumptions of Proposition 4 hold. Then

$$\|f_\varepsilon - \chi_{u_\varepsilon}\|_{L^1((0,T)\times\Omega\times\mathbb{R}_v)} \leqslant \varepsilon m$$

where m is some constant only depending on f_0 and $\tilde{f}$.

Proof of Proposition 5. The proof of Proposition 5 follows from the equality

$$\|f_\varepsilon - \chi_{u_\varepsilon}\|_{L^1((0,T)\times\Omega\times\mathbb{R}_v)} = \varepsilon \left\| \frac{\partial f_\varepsilon}{\partial t} + a(v)\cdot\nabla_x f_\varepsilon \right\|_{M((0,T)\times\Omega\times\mathbb{R}_v)}$$

as well as (16) and the BV estimates of (f_ε) established in Proposition 4.

Lemma 6. Let ϕ be a nonnegative test function in $C_0^1([0,T]\times\bar{\Omega})$. Then

$$\int_0^T \int_{\Omega\times\mathbb{R}} (\chi_{u_\varepsilon} - f_\varepsilon)\,\mathrm{sgn}(f_\varepsilon - \chi_\ell)\,\phi\,dx\,dt\,dv \leqslant 0.$$

Proof of Lemma 6.

$$
\begin{aligned}
&\int_{\mathbb{R}} (\chi_{u_\varepsilon} - f_\varepsilon)\operatorname{sgn}(f_\varepsilon - \chi_\ell)\, dv \\
&\quad = \int_{\mathbb{R}} [(\chi_{u_\varepsilon} - \chi_\ell) - (f_\varepsilon - \chi_\ell)]\operatorname{sgn}(f_\varepsilon - \chi_\ell)\, dv \\
&\quad \leqslant \int_{\mathbb{R}} |\chi_{u_\varepsilon} - \chi_\ell|\, dv - \int_{\mathbb{R}} |f_\varepsilon - \chi_\ell|\, dv \\
&\quad = |u_\varepsilon - l| - \int_{\mathbb{R}} |f_\varepsilon - \chi_\ell|\, dv \\
&\quad = \left|\int_{\mathbb{R}} (f_\varepsilon - \chi_\ell)\, dv\right| - \int_{\mathbb{R}} |f_\varepsilon - \chi_\ell|\, dv \leqslant 0 \quad \blacksquare
\end{aligned}
$$

Then Lemma 5 follows from the non-negativity of the test function $\phi(t, x)$.

Proof of Theorem 1. First Step. The Entropy Inequality. The proof follows the lines of ref. 13. The solution f_ε to (10) satisfies also

$$
\begin{aligned}
&\frac{\partial}{\partial t}(f_\varepsilon - \chi_\ell) + a(v)\cdot\nabla_x(f_\varepsilon - \chi_\ell) \\
&\qquad = \frac{1}{\varepsilon}(\chi_{u_\varepsilon} - f_\varepsilon), \qquad (t, x, v) \in (0, T)\times\Omega\times\mathbb{R} \\
&f_\varepsilon(t, x, v) = \tilde{f}(t, x, v), \qquad t \in (0, T), \quad (x, v) \in \Gamma^-
\end{aligned}
$$

for any real number ℓ. For a nonnegative test function $\varphi = \varphi(t, x, v)$, with compact support in $]0, T[\times\mathbb{R}_+\times\mathbb{R}^{N-1}$, the weak formulation of the kinetic problem is

$$
\begin{aligned}
&-\int_0^T\int_{\Omega\times\mathbb{R}} [(\partial_t + a(v)\cdot\nabla_x)\varphi](f_\varepsilon - \chi_\ell) + \int_0^T\int_{\Gamma^-} a(v)\cdot n(\tilde{f} - \chi_\ell)\varphi \\
&\quad + \int_0^T\int_{\Gamma^+} a(v)\cdot n(f_\varepsilon - \chi_\ell)\varphi = \frac{1}{\varepsilon}\int_0^T\int_{\Omega\times\mathbb{R}} (\chi_{u_\varepsilon} - f_\varepsilon)\varphi, \qquad \ell \in \mathbb{R}.
\end{aligned}
$$

Define the test function φ by $\varphi(t, x, v) = \operatorname{sgn}^\nu(f_\varepsilon - \chi_\ell)\,\psi(t, x)$, where ψ is a nonnegative test function and sgn^ν a regularization of the *sign* function, such that

$$
x \operatorname{sgn}^\nu(x) \geqslant 0, \qquad x \in \mathbb{R}.
$$

Taking into account that

$$\begin{aligned}[(\partial_t+a(v)\cdot\nabla_x)\varphi](f_\varepsilon-\chi_\ell) &= [(\partial_t+a(v)\cdot\nabla_x)\psi][\operatorname{sgn}^\nu(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell)\\ &\quad+[(\partial_t+a(v)\cdot\nabla_x)(f_\varepsilon-\chi_\ell)]\,\psi(\operatorname{sgn}^\nu)'\,(f_\varepsilon-\chi_\ell)\\ &= [(\partial_t+a(v)\cdot\nabla_x)\psi][\operatorname{sgn}^\nu(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell)\\ &\quad+\frac{\chi_{u_\varepsilon}-f_\varepsilon}{\varepsilon}\psi[(\operatorname{sgn}^\nu)'\,(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell)\end{aligned}$$

we get

$$\begin{aligned}&-\int_0^T\int_{\Omega\times\mathbb{R}}[(\partial_t+a(v)\cdot\nabla_x)\psi][\operatorname{sgn}^\nu(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell)\\ &\quad+\int_0^T\int_{\Gamma^-}a(v)\cdot n[\operatorname{sgn}^\nu(\tilde{f}-\chi_\ell)](\tilde{f}-\chi_\ell)\psi\\ &=-\int_0^T\int_{\Gamma^+}a(v)\cdot n[\operatorname{sgn}^\nu(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell)\psi\\ &\quad+\int_0^T\int_{\Omega\times\mathbb{R}}\frac{\chi_{u_\varepsilon}-f_\varepsilon}{\varepsilon}[\operatorname{sgn}^\nu(f_\varepsilon-\chi_\ell)]\psi\\ &\quad+\int_0^T\int_{\Omega\times\mathbb{R}}\frac{\chi_{u_\varepsilon}-f_\varepsilon}{\varepsilon}\psi[(\operatorname{sgn}^\nu)'\,(f_\varepsilon-\chi_\ell)](f_\varepsilon-\chi_\ell).\end{aligned}$$

The first term in the righty-hand side is nonpositive since $a(v)\cdot n\geqslant 0$ on Γ^+ and $x\operatorname{sgn}^\nu(x)\geqslant 0$, $x\in\mathbb{R}$. Passing to the limit with the regularization parameter ν, the last term vanishes, so that

$$\begin{aligned}&-\int_0^T\int_{\Omega\times\mathbb{R}}[(\partial_t+a(v)\cdot\nabla_x)\psi]\,|f_\varepsilon-\chi_\ell|+\int_0^T\int_{\Gamma^-}a(v)\cdot n\,|\tilde{f}-\chi_\ell|\,\psi\\ &\leqslant\int_0^T\int_{\Omega\times\mathbb{R}}\frac{\chi_{u_\varepsilon}-f_\varepsilon}{\varepsilon}\operatorname{sgn}(f_\varepsilon-\chi_\ell)\psi.\end{aligned}$$

By Lemma 6, the term in the right-hand side is nonpositive. And so,

$$-\int_0^T\int_{\Omega\times\mathbb{R}}[(\partial_t+a(v)\cdot\nabla_x)\psi]\,|f_\varepsilon-\chi_\ell|+\int_0^T\int_{\Gamma^-}a(v)\cdot n\,|\tilde{f}-\chi_\ell|\,\psi\leqslant 0.$$

It follows from Proposition 5 that

$$-\int_0^T \int_{\Omega\times\mathbb{R}} [(\partial_t + a(v)\cdot\nabla_x)\psi]\, |\chi_{u_\varepsilon} - \chi_\ell|$$
$$+\int_0^T \int_{\Gamma^-} a(v)\cdot n\, |\tilde{f} - \chi_\ell|\, \psi \leqslant R_\varepsilon(\psi)$$

with $\lim_{\varepsilon\to 0} R_\varepsilon(\psi) = 0$. And so,

$$-\int_0^T \int_\Omega |u_\varepsilon - \ell|\, \partial_t \psi - \int_0^T \int_\Omega \operatorname{sgn}(u_\varepsilon - \ell)(A(u_\varepsilon) - A(\ell))\cdot\nabla_x \psi$$
$$+\int_0^T \int_{\Gamma^-} a(v)\cdot n\, |\tilde{f} - \chi_\ell|\, \psi \leqslant R_\varepsilon(\psi).$$

Passing to the limit when $\varepsilon \to 0$ in the previous equation leads to

$$-\int_0^T \int_\Omega |u - \ell|\, \partial_t \psi + \operatorname{sgn}(u - \ell)(A(u) - A(\ell)\cdot\nabla_x \psi$$
$$+\int_0^T \int_{\Gamma^-} a(v)\cdot n\, |\tilde{f} - \chi_\ell|\, \psi \leqslant 0 \tag{16}$$

for any test function $\psi(t, x)$, with ψ nonnegative and supported in $]0, T[\times \Omega$. Note that Helly's theorem on BV function makes possible this limit (up to a subsequence) since we have uniform bounds of (u_ε) in $BV((0, T)\times\Omega)$.

Boundary Condition. By the inequality (16), u satisfies the conservation law and its entropy inequality in Ω. Moreover, on $\partial\Omega$,

$$\operatorname{sgn}(u - \ell)(A(u) - A(\ell))\cdot n - \int_{\Gamma^-} a(v)\cdot n\, |\tilde{f} - \chi_\ell|\, dv \geqslant 0, \qquad \ell\in\mathbb{R} \tag{K}$$

Let us refer to this last inequality as the (K) condition. In the particular case of an equilibrium datum at the boundary, i.e., when $\tilde{f}(t, x, v) := \chi_{w(t,x)}(v)$ for some function $w(t, x)$, the condition (K) becomes

$$\operatorname{sgn}(u - \ell)(A(u) - A(\ell))\cdot n$$
$$-\operatorname{sgn}(w - \ell)((A\cdot n)^-(w) - (A\cdot n)^-(\ell)) \geqslant 0, \qquad \ell\in\mathbb{R} \tag{KE}$$

Finally, the uniqueness of u follows from the result of uniqueness in the Bardos, Leroux and Nedelec context.[1]

5. THE KINETIC BOUNDARY CONDITION VERSUS THE (BLN) CONDITION

5.1. The Equilibrium Case

We consider the case where $\tilde{f} = \chi_w$, for some function w.

Proposition 7. (i) If u satisfies the (BLN) condition (3), then u also satisfies the (KE) condition.

(ii) Conversely, if u satisfies the (KE) condition, and if $(A \cdot n)$ is non degenerated in the sense that $(A \cdot n)''$ keeps a constant sign, then u also satisfies the (BLN) condition (3).

Proof of Proposition 7. For simplicity let us denote by

$$g(z) := (A \cdot n)(z), \qquad g^-(z) := (A \cdot n)^-(z), \qquad g^+(z) := g(z) - g^-(z)$$

First Step. Recall that the (BLN) condition (3) is equivalent to

$$\operatorname{sgn}(u - l)(g(u) - g(\ell)) \geqslant 0, \qquad \ell \in I(u, w) \tag{17}$$

where $I(u, w) := \{z \in \mathbb{R}; z = \theta u + (1 - \theta) w, 0 \leqslant \theta \leqslant 1\}$.

Second Step. Let us prove that the (KE) condition is equivalent to

$$\operatorname{sgn}(u - l)(g(u) - g(\ell)) \geqslant \operatorname{sgn}(w - \ell)(g^-(w) - g^-(\ell)), \qquad \ell \in I(u, w) \tag{18}$$

To this end, we only need to prove that taking $\ell \notin I(u, w)$ in (KE) does not bring more information than (18). Let us first consider $\ell \geqslant \max(u, w)$ in (KE), so that

$$g^-(w) - g^-(\ell) \geqslant g(u) - g(\ell), \qquad \ell \geqslant \max(u, w)$$

This is equivalent to

$$g^-(w) + g^+(\ell) \geqslant g(u), \qquad \ell \geqslant \max(u, w).$$

Taking into account that g^+ is non decreasing, this comes back to

$$g^-(w) + g^+(\max(u, w)) \geqslant g(u).$$

Analogously, for $\ell \leqslant \min(u, w)$,

$$g^-(w) + g^+(\min(u, w)) \leqslant g(u).$$

And so, the condition (KE) for $\ell \notin I(u, w)$ reduces to

$$\operatorname{sgn}(u-w)(g(u)-g(w)) \geqslant 0.$$

This last inequality is also obtained by taking $\ell = w$ in (19), which finally proves that the (KE) condition is equivalent to (18).

Third Step. The (BLN) condition implies the (KE) condition, since

$$\operatorname{sgn}(w-\ell)(g^{-}(w)-g^{-}(\ell)) \leqslant 0$$

g^{-} being non increasing.

Fourth Step. Let us prove that the (KE) condition implies the (BLN) condition, under the additional requirement that g' is either non decreasing or non increasing. We first look for $u \geqslant w$ satisfying (18).

Taking $\ell = w$ in (18), we get that $g(u) \geqslant g(w)$. Note also that the (BLN) condition reduces then to $g(u) \geqslant g(\ell)$ for any ℓ such that $w \leqslant \ell \leqslant u$.

- Let us suppose that $g'(w) \leqslant 0$, and let

$$w_1 = \sup_{z \geqslant w} \{z;\ g(z) = g(w)\}$$

Since g is non-degenerated, if $w_1 > w$, we have $g'(u) \geqslant 0$ for any $u \geqslant w_1$ and the (KE) condition is only true for any $u \geqslant w_1$. Note that in this case the (BLN) condition is also true. If $w_1 = w$, $g(u) < g(w)$ for any $u > w$, so that the (KE) condition and the (BLN) condition are both equivalent to $u = w$.

- We suppose now that $g'(w) > 0$. Either $g'(u) \geqslant 0$, for any $u \geqslant w$ and the conditions (KE) and (BLN) both reduce to $u \geqslant w$. Or g has a maximum $g_1 < +\infty$ on $[w, +\infty[$. Let $w_1 < +\infty$ be the greatest real $w_1 > w$, such that $g(w_1) = g_1$. Then $g'(u) \geqslant 0$, for any $u \in [w, w_1]$. It follows that $g^{-}(w_1) = g^{-}(w)$. Clearly the (BLN) condition reduces to $u \in [w, w]$. Consequently the (KE) condition is true for $u \in [w, w_1]$. If there were some $u > w_1$ satisfying the (KE) condition, then for $\ell = w_1$, $g(u) \geqslant g(w_1)$. Moreover, $g(u) \geqslant g(w)$. This contradicts the inequality $g'(z) \leqslant 0$ for any $z > w_1$. It follows that the (KE) condition implies the (BLN) condition for $u \geqslant w$. The case where $u < w$ can be analyzed similarly.

Remark 8. The hypothesis on the non degeneracy of $(A \cdot n)$ is essential in our proof. Otherwise, the set given by the (KE) condition can be strictly bigger than the set given by the (BLN) condition.

Remark 9. The same result also holds and the proof is simpler if $(A \cdot n)$ is either non-decreasing or non-increasing.

We can state the following corollary to Theorem 1.

Corollary 10. Assume that the boundary data $\tilde{f}$ is at equilibrium, i.e., $\tilde{f}(t, x, v) = \chi_{w(t,x)}(v)$ for some function w, that the assumptions of Proposition 3 hold, and $(A \cdot n)$ is non-degenerated or non-increasing, or non-decreasing. Then the family (u_ε) converges in $L^\infty(0, T; L^1_{loc}(\Omega))$ to the entropy solution of the mixed problem (2) with the (BLN) condition for the data w.

Proof of Corollary 10. The proof consists in applying Theorem 1 together with the results of Proposition 7. We conclude easily since the solution of (2) is unique. ∎

5.2. A Case far from Equilibrium

In this section, we prove the second part of Theorem 1, stated in the introduction, by discussing the case of the following particular boundary data on the kinetic side

$$\tilde{f}(t, x, v) = \tfrac{1}{2}(\chi_{w_1}(v) + \chi_{w_2}(v)) \tag{19}$$

where w_1 and w_2 are two positive constants such that $w_1 < w_2$. Let us exactly compute the boundary condition. We use the same notations as in the previous section ($g(z) := (A \cdot n)(z)$ and $g^-(z) := (A \cdot n)^-(z)$). A direct computation gives

$$\int_{\Gamma^-} a(v) \cdot n \, |\tilde{f} - \chi_\ell| \, dv$$

$$= \int \min(0, g'(v)) \, |\tfrac{1}{2}(\chi_{w_1} + \chi_{w_2}) - \chi_\ell| \, dv$$

$$= G(w_1, w_2, l) := \begin{cases} g^-(l) - \frac{1}{2}(g^-(w_1) + g^-(w_2)) & \text{if } \ \ell > w_2 \\ \frac{1}{2}(g^-(w_2) - g^-(w_1)) & \text{if } \ w_2 \geqslant \ell \geqslant w_1 \\ \frac{1}{2}(g^-(w_1) + g^-(w_2)) - g^-(l) & \text{if } \ \ell < w_1 \end{cases}$$

Hence the boundary condition (K) becomes

$$\operatorname{sgn}(u - \ell)(g(u) - g(\ell)) \geqslant G(w_1, w_2, \ell), \qquad \ell \in \mathbb{R} \tag{20}$$

Proposition 11. If $(A \cdot n)$ is non degenerated, or non increasing or non decreasing, then the kinetic boundary condition (21) is equivalent to the (BLN) condition for the data $\tilde{w}$ defined by

$$\int_{a(v)\cdot n(x)<0} a(v)\cdot n(x)\,\chi_{\tilde{w}}(v)\,dw = \int_{a(v)\cdot n(x)<0} a(v)\cdot n\tilde{f}(v)\,dv$$

Remarks. This former equality of inward fluxes, defining the state $\tilde{w}$, is used in numerical studies when coupling kinetic and fluid equations.[3]

In general $\tilde{w}$ is different from the average of $\tilde{f}(x, t, v)$,

$$\tilde{w} \neq \int \tilde{f}(t, x, v)\,dv = \tfrac{1}{2}(w_1 + w_2)$$

Corollary 12. The initial boundary-value problem

$$\frac{\partial u}{\partial t} + div_x(A(u)) = 0, \qquad (t, x) \in (0, T) \times \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

$$\text{Condition (K)}, \qquad t \in (0, T), \qquad x = (0, y) \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

has a unique solution.

Proof of Corollary 12. The existence of a solution to such a problem has been proven in Section 4. Its uniqueness follows from Proposition 11 and the uniqueness of the solution to the problem

$$\frac{\partial u}{\partial t} + div_x(A(u)) = 0, \qquad (t, x) \in (0, T) \times \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

$$\text{(BLN) for the data } \tilde{w}, \qquad t \in (0, T), \qquad x = (0, y) \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$$

Proof of Proposition 11. Let us denote by

$$g_1^- := g^-(w_1), \qquad g_2^- := g^-(w_2), \qquad \bar{g}^- := \tfrac{1}{2}(g_1^- + g_2^-)$$

First Step. Taking successively $\ell < \min(u, w_1) := m_1$ and $\ell > \max(u, w_2) := m_2$ implies, as in the second step of the proof of Proposition 7, that the (K) condition (21) is equivalent to

$$\text{(i)}\quad \operatorname{sgn}(u-\ell)(g(u)-g(\ell)) \geqslant G(w_1, w_2, \ell), \qquad \ell \in [m_1, m_2]$$

and (21)

$$\text{(ii)}\qquad \bar{g}^- + g^+(m_1) \leqslant g(u) \leqslant \bar{g}^- + g^+(m_2)$$

Let us also remark that $G(w_1, w_2, \ell) \leqslant 0$ for any $\ell \in \mathbb{R}$.

Second Step. We suppose that g is non-decreasing, thus we can take $g^+ = g$ and $g^- = 0$. Hence $G(w_1, w_2, \ell) = 0$ and the (K) condition becomes

$$\operatorname{sgn}(u-\ell)(g(u)-g(\ell)) \geqslant 0, \qquad \ell \in \mathbb{R}$$

which is always true and equivalent to the (BLN) condition for any data $\tilde{w}$. That means that the flow is outward and no boundary condition is needed neither for the (K) case nor for the (BLN) case. Evidently, any data $\tilde{w}$ is solution to

$$g^-(\tilde{w}) = \tfrac{1}{2}(g^-(w_1) + g^-(w_2))$$

i.e.,

$$\int_{a(v)\cdot n<0} a(v)\cdot n\chi_{\tilde{w}}(v)\,dv = \int_{a(v)\cdot n<0} a(v)\cdot n\tilde{f}(v)\,dv.$$

Third Step. We suppose that g is non-increasing, thus we can take $g^+ = 0$ and $g^- = g$. The (BLN) condition for the data w is equivalent to $u = w$. By (21)(ii), $\bar{g}^- = g(u)$. Omitting the special case where g is constant on a set of non null measure in $[w_1, w_2]$, we get $u = g^{-1}(\bar{g}^-)$. And so, unless g is linear in $[w_1, w_2]$, $u = \tilde{w}$ with $\tilde{w} \neq \frac{1}{2}(w_1 + w_2)$. Moreover, (22)(i) comes back to

$$g(u) \geqslant \tfrac{1}{2}(g(w_1) + g(w_2)), \qquad \ell \in [w_1, u]$$

$$g(u) \leqslant \tfrac{1}{2}(g(w_1) + g(w_2)), \qquad \ell \in [u, w_2]$$

so is satisfied. Thus the (K) condition is equivalent to the (BLN) condition for the data $\tilde{w}$. Moreover, $\tilde{w}$ can also be defined by the condition

$$g^-(\tilde{w}) = \tfrac{1}{2}(g^-(w_1) + g^-(w_2))$$

i.e.,

$$\int_{a(v)\cdot n<0} a(v)\cdot n\chi_{\tilde{w}}(v)\,dv = \int_{a(v)\cdot n<0} a(v)\cdot n\tilde{f}(v)\,dv.$$

Fourth Step. We now study the case g convex, with g non-increasing for $\ell \leqslant w^*$ and non-decreasing for $\ell \geqslant w^*$ for some w^*. For simplicity we limit ourselves to the case where g is strictly convex. We can take $g^+(z)=0$ for $z \leqslant w^*$, so that

$$g^-(z) = \int_{w^*}^{z} \min(0, a(v)\cdot n)\, dv + g(w^*), \qquad z \in \mathbb{R}.$$

Let us consider three cases, according to the location of w_1 and w_2 with respect to w^*. Recall that the set of solutions to (BLN) for the data w is $[w^*, +\infty[$ if $w \geqslant w^*$, and $\{w\} \cup [w', +\infty[$ if $w < w^*$, where w' is the solution to

$$g(w') = g(w), \qquad w' > w^*.$$

Fifth Step. g Convex, $w_1 < w_2 \leqslant w^*$. Denote by $\tilde{w}$ and $\tilde{w}'$ the solutions to

$$g(\tilde{w}) = g(\tilde{w}') = \tfrac{1}{2}(g(w_1) + g(w_2)), \qquad w_1 < \tilde{w} < w_2 < w^* < \tilde{w}'.$$

- Let us first look for the states $u \leqslant w^*$ satisfying the condition (K). Then m_1 and m_2 are both smaller than w^*. Thus (21)(ii) is equivalent to $\bar{g}^- = g(u)$, i.e., $u = g^{-1}(\bar{g}^-) = \tilde{w}$. Then (22)(ii) holds. And so, $\tilde{w}$ is the only solution to (K) smaller than w^*.

- Let us now look for the states $u > w^*$ satisfying the condition (K). Then m_1 and m_2 are given respectively by $m_1 = w_1$ and $m_2 = u$. The condition (21)(ii) is equivalent to

$$g(u) \geqslant \tfrac{1}{2}(g(w_1) + g(w_2))$$

i.e., $u \geqslant \tilde{w}'$. The condition (22)(i) is then satisfied. Hence, the set of solutions to (K) bigger than w^* is $[\tilde{w}', +\infty[$.

Consequently, the set of solutions to (K) in the case where $w_1 < w_2 \leqslant w^*$ is the set of solutions to (BLN) for the data $\tilde{w}$. Moreover, the definition of $\tilde{w}$ by

$$g(\tilde{w}) = \tfrac{1}{2}(g w_1) + g(w_2)), \qquad w_1 < \tilde{w} < w_2$$

can also be expressed by

$$g^-(\tilde{w}) = \tfrac{1}{2}(g^-(w_1) + g^-(w_2))$$

i.e., the equality of inward fluxes

$$\int_{a(v)\cdot n<0} a(v)\cdot n\chi_{\tilde{w}}(v)\,dv = \int_{a(v)\cdot n<0} a(v)\cdot n\widetilde{f}(v)\,dv.$$

Sixth Step. g Convex, $w_1 \leqslant w^* \leqslant w_2$. In this case,

$$G(w_1, w_2, \ell) = \tfrac{1}{2}(g(w^*) - g(w_1)), \qquad \ell \geqslant w_1$$

$$G(w_1, w_2, \ell) = \tfrac{1}{2}(g(w_1) + g(w^*)) - g(\ell), \qquad \ell < w_1.$$

Denote by $\tilde{w}$ and $\tilde{w}'$ the solutions to

$$g(\tilde{w}) = g(\tilde{w}') = \tfrac{1}{2}(g(w_1) + g(w^*)), \qquad w_1 < \tilde{w} < w^* \leqslant \tilde{w}'.$$

• Let us first look for $u \geqslant w^*$ satisfying the condition (K). The condition (22)(ii) is then

$$g(u) \geqslant \tfrac{1}{2}(g(w_1) + g(w^*))$$

i.e., $u \geqslant \tilde{w}'$. The condition (22)(i) is then satisfied.

• Let us now look for $u \in [w_1, w^*]$ satisfying the condition (K). Taking successively $w_1 < \ell < u$ and $u < \ell < w_2$ in (21)(i), we get $g(u) = \frac{1}{2}(g(w_1) + g(w^*))$, i.e., $u = \tilde{w}$. Moreover, the condition (22)(ii) is satisfied. Hence $u = \tilde{w}$ is the only solution to (K) in $[w_1, w^*]$.

• It remains to look for $u < w_1$ satisfying the condition (K). Taking $\ell = w^*$ in (21)(i), we get $g(u) \leqslant \frac{1}{2}(g(w_1) + g(w^*))$, which is not possible since

$$\tfrac{1}{2}(g(w_1) + g(w^*)) < g(w_1) < g(u).$$

Thus there is no solution $u < w_1$.

Hence, the set of solutions to (K) in the case where $w < w^* \leqslant w_2$ is $\{\tilde{w}\} \cup [\tilde{w}', +\infty[$, i.e., the set of solutions to (BLN) for the data $w = \tilde{w}$. Moreover, the definition of $\tilde{w}$ by

$$g(\tilde{w}) = \tfrac{1}{2}(g(w_1) + g(w^*)), \qquad w_1 < \tilde{w} < w^*$$

can also be expressed by

$$g^-(\tilde{w}) = \tfrac{1}{2}(g^-(w_1) + g^-(w_2))$$

i.e., the equality of the inward fluxes

$$\int_{a(v)\cdot n<0} a(v)\cdot n\chi_{\tilde{w}}(v)\,dv = \int_{a(v)\cdot n<0} a(v)\cdot n\tilde{f}(v)\,dv.$$

Seventh Step. g Convex, $w^* \leqslant w_1 < w_2$. A similar computation as in the previous steps proves easily that the set of solutions to the (K) condition is $[w^*, +\infty[$, which is the set of solutions to the (BLN) condition for a data $\tilde{w} \geqslant w^*$. Such data $\tilde{w} \geqslant w^*$ can also be defined by the condition

$$g^-(\tilde{w}) = g(w^*)$$

i.e.,

$$\int_{a(v)\cdot n<0} a(v)\cdot n\chi_{\tilde{w}}(v)\,dv = \int_{a(v)\cdot n<0} a(v)\cdot n\tilde{f}(v)\,dv.$$

Eight Step. g Concave. We do not detail this case which has to be studied in 3 separate cases according to the location of w_1 and w_2 to the possible maximum of g, w^*. The conclusions are similar to those of the convex case. ▮

6. CONCLUSION

In this paper we have studied the fluid limit of some kinetic model for scalar conservation laws. In the case of non-degenerated flux and kinetic data at equilibrium at the boundary, we recover the classical condition of Bardos, Leroux and Nedelec. If the data at the boundary is not at equilibrium we obtain new informations. This is connected with more general studies of boundary layers (see ref. 7).

ACKNOWLEDGMENT

We thank M. Rascle for suggesting to make appear the equality of inward fluxes as a definition of the states $\tilde{w}$.

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